# HW #6: Hints

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October 4, 2000

## 1 Period of Oscillation in a Power-Law Potential

First of all, since the problem only asks you to find the period of oscillation, you don't need to find $x(t)$. In fact, you can't. Just use energy conservation to calculate $x_{max}$ and $x_{min}$ for the particle, and to find $\dot{x}(x)$. Then use the fact that $dt = \frac{dt}{dx}$ (how can you find $\frac{dt}{dx}$?) to set up an integral for part of the period (it's easier to do a fraction of the whole period than the whole thing at once; just calculate the total period from this particular fraction.)

You'll have to do a substitution to get the integral into the following form:

$$\int_0^1 \frac{y^{\frac{1}{n}-1}dy}{\sqrt{1-y}}$$

Of course, everyone learned that integral in junior high school. Not. It's a $\beta$ function, with the property

$$\beta(p,q) \equiv \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

If you don't know what the $\Gamma$ function is, it's much like the factorial function – specifically, $\Gamma(n+1)=n!$ for all integer $n > 0$. However, the $\Gamma$ function is also defined at non-integer $n$; it's defined as:

$$\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}dt$$

Who would have thought that would give you a factorial? Anyway, you don't need to know all that for the problem, but it's good to know about Gamma functions. The only thing you need to know is that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

## 2 Underdamped Oscillator

You should find the Green's function for the underdamped oscillator in the lecture notes. Recall that because the oscillator is linear,

$$D\sum_i x_i(t) = \sum_i \frac{F_i}{m}(t)$$

if $x_i(t)$ is the solution to the driving term $\frac{F_i}{m}(t)$. Thus, if you have the solution to a delta function driving term:

$$D\left[G(t-t')\right] = \delta(t-t')$$

then by writing an arbitrary driving term as $F(t) = \int_{-\infty}^{\infty} F(t')\delta(t-t')dt'$, we can write the solution as

$$x(t) = \int_{-\infty}^{t} F(t')G(t-t')dt'$$

The reason for not letting the upper limit of integration go to $\infty$ is to deftly include the requirement that $G(t) = 0$ for $t < 0$. That makes sense, right? – it just means that a driving force that *happens* at $t = 0$ cannot cause motion at $t < 0$. Alternatively, if $t'$ is later than $t$, then a driving force at $t'$ can't cause motion at $t$. It's easier to change the limit of integration than to explicitly make $G(t < 0) = 0$ – look in the notes and you'll see that $G(t-t')$ has a nice functional form as long as you *implicitly* add the assumption that $G(t < 0) = 0$.

Finally, once you've set up the integral, I recommend you do a slight substitution to set the origin of time $t = 0$ to a convenient locale. From there it's just math (though the answer is rather heinous). Once you get an answer, you may wish to check the answer in the solution set, and fix any mistakes you find (better than me correcting them when I grade, 'cause this way you get your feedback immediately!)

## 3 Perturbation Theory for a Nonlinear Oscillator

Your strategy is as follows (this is the fundamental method of "perturbation theory" in any branch of physics). Assume that $\lambda$ is a small number, so that $\lambda^2$ is *much* smaller than $\lambda$. Then, since you don't know the solution, you can quite generally write it as a *power series* in $\lambda$ – that is,

$$x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \ldots$$

but you immediately decide to ignore all terms with more than one power of $\lambda$ throughout the solution (that's what's meant by "accurate to first order"), so $x(t) = x_0(t) + \lambda x_1(t)$.

Now plug it into the differential equation, expand things out, and ditch all the $\lambda^2$ and $\lambda^3$ terms (etc, etc). Once you've got your equation to 1st order in $\lambda$, you can separate into two equations, one of which has terms with no powers of $\lambda$ and one of which has terms proportional to $\lambda$, on the philosophy that this has to be valid no matter what the value of $\lambda$ is. Think of it as similar to taking a complex equation with real and imaginary parts and separately solving for the real and imaginary parts.

Anyway, one of those equations will give you a solution for $x_0(t)$. (Don't forget to match initial conditions!) You can take that solution and plug it into the other equation. You get another equation that's familiar, but with a weird driving term. Specifically, it has an absolute value in it! Just consider separately the first half-period where the term in $|f(t)|$ is equal to $f(t)$ and the second half-period where $|f(t)|$ is equal to $-f(t)$.

You may want to guess a particular solution of the form $x_{1p}(t) = A + B\cos 2\omega_0 t$. Then add the homogenous solution $x_{1h}(t)$ and match boundary conditions. That should give you $x_1(t)$ for the first half-period... now do it for the *other* half-period when the driving force changes because of the absolute value.

When matching boundary conditions, use the fact that $x$ and $\dot{x}$ must be continuous if there is no infinite force!

Finally, when you've done all this for the 2nd half-period, see what you end up with at $t = 2\pi\omega_0$. See how this frees you from solving the problem over and over again for each and every succeeding half-period. *Hint (sub-hint?)*: the oscillator is damped.

## 4 Particles w/Gravitational Force

First of all, as always occurs with central-force problems, you'll want to view this as an effective one-body problem – $r$ is the distance between an "effective particle" of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$. You should be able to find the period of the orbit in terms of the radius $R$ of the orbit, using a simple 7A argument about uniform circular motion.

If you're confused by the statement of the problem, then rest assured that when the particles are "stopped in their orbits," they are instantaneously placed *at rest*: $v = 0$. So you can use energy conservation to find $\dot{r}(r)$ as in problem (1), and use a very similar integration technique (remember the

$\beta$ function?) to find $T_{collision}$. To obtain the final answer, use the values of the $\Gamma$ function that I gave you in problem (1), together with the fact (generalized from the factorial function) that

$$\frac{\Gamma(x+1)}{\Gamma(x)} = x$$

# 5 Space Explosions!

This is a pretty straightforward problem – I don't think you'll need much help. However, you may want to consider the implications of the virial theorem (see notes) for the relative kinetic energy of a nose cone in circular and parabolic orbits. What does this imply for the change in momentum of the nose cone? and therefore for the change in momentum of the service module, which (I remind you) falls "directly," or straight, into the sun. Use conservation of momentum to find the relative masses.

# 6 Central Power Law Potential

First of all, try playing around with the equations that you have in order to write the total energy of the particle as a function of $r$ only. Specifically, you gotta write $\dot{r}$ and $\dot{\theta}$ in terms of $r$. If you can't get it, here's a more specific hint.

Try writing the orbit (a circle that passes through the origin) in polar coordinates. Don't worry about time, just parametrize the curve as $r(\theta)$. Remember that everything that is usually conserved ($\vec{p}$, $\vec{L}$, and $E$) is conserved, and use the conservation laws together with the time derivative of the orbit equation (i.e., $r(\theta)$) to write down the same sort of $\dot{r}(r)$ relation we've already used twice.

Now, you can write the kinetic energy in terms of $r$ and you can also write the potential energy in terms of $r$ (you're given $F$, right? and $F = -\frac{\partial U}{\partial r}$, right?). So just make sure that the total energy is (because it's conserved) a constant that doesn't depend on $r$ – which means that any powers of $r$ that you have in $E$ must (a) be the same, and (b) cancel each other out term by term.

# 7 Spaceship Jitters

Note that only the direction of the velocity vector changes when the engines fire.

How does $E_f$, the energy after firing, compare to $E_i$, the energy before? What about $L_f$ versus $L_i$ (hint: write the new velocity in polar-coordinate vector form)?

Use (7.12) and (7.10) from the notes to use your information about the change in $L$ and $E$ to find the new eccentricity.

# 8 Boing, Boing

## (a)

Set this up as a Lagrangian problem, using generalized coordinates based on the position of the puck on the table (that determines the height of the weight). What's conserved here (hint, look at the Lagrangian, and the E-L equations if necessary)?

When you calculate the E-L equation for $r$, you'll get an equation of motion for $\ddot{r}$. This gets even simpler for a circular orbit – think about the behavior of $r$ for a circular orbit. The trick here is to ignore $\theta$ and imagine perturbing a circular orbit of radius $R$ by a small amount $\epsilon$ – that is, $r = R + \epsilon$. Use the E-L equation to find the equation of motion for $\epsilon$, and simplify it by a Taylor expansion of the nonlinear term for $\epsilon << R$. Since we're talking oscillations, you had better end up with an equation for $\epsilon$ like $\ddot{\epsilon} \propto -\epsilon$. That gives you the frequency of perturbation.

## (b)

If the ratio of two frequencies is irrational, then the orbit can never close. If you use this hint, I'd like to see a creative (well, some kind of) explanation for *why* this is true – not just a statement that it is true. Oh... and you should certainly show that my statement applies to this case!